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Integration Technique for Singularly Perturbed Delay Differential Equations

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Abstract— In this paper, we have described a numerical integration technique for solving singularly perturbed delay differential equations. The second order singularly perturbed boundary value problem is transformed into an asymptotically equivalent first order neutral differential equation. Then numerical integration and linear interpolation is used to get the tri-diagonal system. Discrete invariant imbedding algorithm is used to solve this tridiagonal system. The error analysis of the technique is discussed. To demonstrate the technique and affect of the delay argument in the layer, we have implemented the technique on several test examples.

Index Terms— Singularly perturbed delay differential equation, Boundary layer, Trapezoidal rule, Linear interpolation, Tridiagonal system, maximum absolute error.

1 Introduction

Any ordinary differential equation, in which the highest derivative is multiplied by a small parameter and involving at least one delay term is called singularly perturbed delay differential equation. In recent years, there has been a growing interest in the numerical treatment of such differential equations. The boundary value problems of delay differential equations are ubiquitous in the variational problems in control theory [3]. Lange and Miura [7, 8] gave an asymptotic approach for a class of boundary-value problems for linear second-order differential-difference equations in which the highest order derivative is multiplied by small parameter and shows the effect of very small shifts on the solution and pointed out that they drastically affect the solution and therefore cannot be neglected. Kadalbajoo and Sharma [6] presented a numerical approach to solve singularly perturbed differential-difference equation, which contains only negative shift in the differentiated term. In this method authors present a numerical method composed of a standard upwind finite difference scheme on a special type of mesh shifts are either $o(\varepsilon)$ or $O(\varepsilon)$. Pratima Rai and Sharma [10] presented a numerical method for singularly perturbed delay differential equation with turning points. Reddy et. al. [11] presented a numerical integration of a class of singularly perturbed delay differential equations with small shift, where delay is in differentiated term.

In In this paper, we have described a numerical integration technique for solving singularly perturbed delay differential equations. The second order singularly perturbed boundary value problem is transformed into an asymptotically equivalent first order neutral differential equation. Then numerical integration and linear interpolation is used to get the tri-diagonal system. Discrete invariant imbedding algorithm is used to solve this tri-diagonal system. The error analysis of the technique is discussed. To demonstrate the technique and affect of the delay argument in the layer, we have implemented the technique on several test examples.

2. Description of the Method

Consider a singularly perturbed delay differential equation

$$\varepsilon y''(x) + a(x)y'(x-\delta) + b(x)y(x) = f(x) \qquad (1)$$

on $0 < x < 1,\ 0 < \delta << 1$, with

$$y(x) = \phi(x), \quad -\delta \le x \le 0, \qquad (2)$$

$$y(1) = \gamma \qquad (3)$$

where $a(x)$, $b(x)$, $f(x)$ are smooth functions, γ is a constant and δ is the delay. For the function $y(x)$ be a smooth solution to the problem (1), it must satisfy: boundary value problem be continuous on [0, 1] and be continuously differentiable on (0, 1).

2.1. Layer on the left side

Here, we consider the case $a(x) \ge M > 0, \forall x \in [0,1]$, M being positive constant. In this case the solution of the boundary value problem exhibits boundary layer behaviour on the left side of the interval [0, 1] i.e., at $x = 0$.

In this section we construct a numerical scheme for solving the boundary value problem based on numerical integration for the case when the solution of the problem exhibits a layer on the left side.

By using Taylor Series expansion in the neighbourhood of the point x, we have

$$y'(x-\varepsilon) \approx y' - \varepsilon y'' \ \text{ and } \ y'(x+\varepsilon) \approx y' + \varepsilon y''$$

Therefore,

$$\varepsilon y'' = \frac{y'(x+\varepsilon) - y'(x-\varepsilon)}{2}$$

Substituting the above equation in (1), it reduces to an asymptotically equivalent first order neutral differential equa-

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tion

$$y'(x+\varepsilon)-y'(x-\varepsilon)=p(x)y'(x-\delta)+q(x)y(x)+r(x) \qquad (4)$$

where $p(x)=-2a$, $q(x)=-2b$, $r(x)=2f(x)$

The transition from equation (1) to equation (4) is admitted, because of the condition that ε is small. This replacement is significant from the computational point of view. Further details on the validity of this transition can be found [5, 9].

Now divide the interval [0, 1] into N equal subintervals of mesh size $h=1/N$ so that $x_i=ih,\ i=0,1,2,\ldots,N.$
Integrating eq. (4) with respect to x from x_i to x_{i+1}, we get

$$\int_{x_i}^{x_{i+1}} y'(x+\varepsilon)dx-\int_{x_i}^{x_{i+1}} y'(x-\varepsilon)dx=$$

$$\int_{x_i}^{x_{i+1}}[p(x)y'(x-\delta)dx+q(x)y(x)+r(x)]dx$$

$$y(x_{i+1}+\varepsilon)-y(x_i+\varepsilon)-y(x_{i+1}-\varepsilon)+y(x_i-\varepsilon)=$$

$$[p(x)y(x-\delta)]_{x_i}^{x_{i+1}}-\int_{x_i}^{x_{i+1}} p'(x)y(x-\delta)dx$$

$$+\int_{x_i}^{x_{i+1}}[q(x)y(x)+r(x)]dx$$

$$2\varepsilon y'_{i+1}-2\varepsilon y'_i=p_{i+1}y(x_{i+1}-\delta)-p_i y(x_i-\delta)$$

$$-\int_{x_i}^{x_{i+1}} p'(x)y(x-\delta)dx+\int_{x_i}^{x_{i+1}} q(x)y(x)+r(x)dx$$

By using the Trapezoidal rule to evaluate the integral approximation, we get

$$2\varepsilon y'_{i+1}-2\varepsilon y'_i=p_{i+1}y(x_{i+1}-\delta)-p_i y(x_i-\delta)-\frac{h}{2}\left(p'_{i+1}y(x_{i+1}-\delta)\right)$$

$$-\frac{h}{2}\left(p'_i y(x_i-\delta)\right)+\frac{h}{2}\left(q_{i+1}y_{i+1}\right)+\frac{h}{2}\left(q_i y_i\right)+\frac{h}{2}r_{i+1}+\frac{h}{2}r_i$$

(5)

Again by means of Taylor series expansion and linear interpolation for $y'(x)$, we get

$$y(x_{i+1}-\delta)\approx y(x_{i+1})-\delta\, y'(x_{i+1})=y_{i+1}-\delta\left(\frac{y_{i+1}-y_i}{h}\right)$$

$$=\left(1-\frac{\delta}{h}\right)y_{i+1}+\frac{\delta}{h}y_i \qquad (6)$$

$$y(x_i-\delta)\approx y(x_i)-\delta y'(x_i)=y_i-\delta\left(\frac{y_i-y_{i-1}}{h}\right)$$

$$=\left(1-\frac{\delta}{h}\right)y_i+\frac{\delta}{h}y_{i-1} \qquad (7)$$

Substituting the equations (6) and (7) in Eq. (5), we get

$$2\varepsilon\left(\frac{y_{i+1}-y_i}{h}\right)-2\varepsilon\left(\frac{y_i-y_{i-1}}{h}\right)$$

$$=\left[p_{i+1}\left(1-\frac{\delta}{h}\right)-\frac{h}{2}p'_{i+1}\left(1-\frac{\delta}{h}\right)+\frac{h}{2}q_{i+1}\right]y_{i+1}$$

$$+\left[p_{i+1}\frac{\delta}{h}-p_i\left(1-\frac{\delta}{h}\right)-\frac{h}{2}p'_{i+1}\left(\frac{\delta}{h}\right)-\frac{h}{2}p'_i\left(1-\frac{\delta}{h}\right)+\frac{h}{2}q_i\right]y_i$$

$$+\left[-p_i\frac{\delta}{h}-\frac{h}{2}p'_i\frac{\delta}{h}\right]y_{i-1}+\frac{h}{2}r_i+\frac{h}{2}r_{i+1}$$

Rearrange the above equation into three recurrence relation, we get

$$\left[p_i\frac{\delta}{h}+\frac{\delta}{2}p'_i+\frac{2\varepsilon}{h}\right]y_{i-1}$$

$$-\left[\frac{4\varepsilon}{h}+p_{i+1}\frac{\delta}{h}-p_i\left(1-\frac{\delta}{h}\right)-\frac{\delta}{2}p'_{i+1}-\frac{h}{2}p'_i\left(1-\frac{\delta}{h}\right)+\frac{h}{2}q_i\right]y_i$$

$$+\left[\frac{2\varepsilon}{h}-p_{i+1}\left(1-\frac{\delta}{h}\right)+\frac{h}{2}p'_{i+1}\left(1-\frac{\delta}{h}\right)-\frac{h}{2}q_{i+1}\right]y_{i+1}=\frac{h}{2}r_i+\frac{h}{2}r_{i+1}$$

The above equation can be written as

$$E_i y_{i-1}-F_i y_i+G_i y_{i+1}=H_i\ ,\ i=1,2,3......n-1 \qquad (8)$$

where

$$E_i=p_i\frac{\delta}{h}+\frac{\delta}{2}p'_i+\frac{2\varepsilon}{h}$$

$$F_i=\frac{4\varepsilon}{h}+p_{i+1}\frac{\delta}{h}-p_i\left(1-\frac{\delta}{h}\right)-\frac{\delta}{2}p'_{i+1}-\frac{h}{2}p'_i\left(1-\frac{\delta}{h}\right)+\frac{h}{2}q_i$$

$$G_i=\frac{2\varepsilon}{h}-p_{i+1}\left(1-\frac{\delta}{h}\right)+\frac{h}{2}p'_{i+1}\left(1-\frac{\delta}{h}\right)-\frac{h}{2}q_{i+1}$$

$$H_i=\frac{h}{2}r_{i+1}+\frac{h}{2}r_i$$

We solve the tridiagonal system (5) using discrete invariant imbedding algorithm.

2.2. Right End boundary Layer

Now for the right - end boundary layer, integrating Eq. (4) with respect to x from x_{i-1} to x_i, we get

$$\int_{x_{i-1}}^{x_i} y'(x+\varepsilon)dx-\int_{x_{i-1}}^{x_i} y'(x-\varepsilon)dx=\int_{x_i-1}^{x_i}[p(x)y'(x-\delta)dx+q(x)y(x)+r(x)]dx$$

$$\left[y(x_i+\varepsilon)-y(x_{i-1}+\varepsilon)\right]-\left[y(x_i-\varepsilon)+y(x_{i-1}-\varepsilon)\right]=[p(x)y(x-\delta)]_{x_{i-1}}^{x_i}$$

$$-\int_{x_{i-1}}^{x_i} p'(x)y(x-\delta)dx+\int_{x_{i-1}}^{x_i}[q(x)y(x)+r(x)]dx$$

$$y_i+\varepsilon\left(\frac{y_{i+1}-y_i}{h}\right)-y_{i-1}-\frac{\varepsilon}{h}(y_i-y_{i-1})-y_i+\varepsilon\left(\frac{y_{i+1}-y_i}{h}\right)$$

$$+y_{i-1}-\frac{\varepsilon}{h}(y_i-y_{i-1})=p_i y(x_i-\delta)-p_{i-1}y(x_{i-1}-\delta)$$

$$-\int_{x_{i-1}}^{x_i}p'(x)y(x-\delta)dx+\int_{x_{i-1}}^{x_i}[q(x)y(x)+r(x)]dx$$

By Using the Trapizoidal rule to evaluate the integral approximation, we get

$$\frac{2\varepsilon}{h}(y_{i+1}-y_i)-\frac{2\varepsilon}{h}(y_i-y_{i-1})=p_i y(x_i-\delta)-p_{i-1}y(x_{i-1}-\delta)$$

$$-\frac{h}{2}\left(p_i' y(x_i-\delta)+p_{i-1}' y(x_{i-1}-\delta)\right)$$

$$+\frac{h}{2}(q_i y_i+q_{i-1}y_{i-1})+\frac{h}{2}(r_i+r_{i-1})$$

$$\frac{2\varepsilon}{h}y_{i+1}-\frac{4\varepsilon}{h}y_i+\frac{2\varepsilon}{h}y_{i-1}=p_i y(x_i-\delta)-p_{i-1}y(x_{i-1}-\delta)-\frac{h}{2}p_i' y(x_i-\delta)$$

$$-\frac{h}{2}p_{i-1}' y(x_{i-1}-\delta)+\frac{h}{2}(q_i y_i)+\frac{h}{2}(q_{i-1}y_{i-1})+\frac{h}{2}(r_i+r_{i-1}) \tag{9}$$

Again by means of Taylor series expansion and then corresponding $y'(x)$ by linear interpolation, we have

$$y(x_{i-1}-\delta)\approx y(x_{i-1})-\delta\, y'(x_{i-1})=y_{i-1}-\delta\left(\frac{y_i-y_{i-1}}{h}\right)$$
$$=\left(1+\frac{\delta}{h}\right)y_{i-1}-\frac{\delta}{h}y_i \tag{10}$$

$$y(x_i-\delta)\approx y(x_i)-\delta y'(x_i)=y_i-\delta\left(\frac{y_{i+1}-y_i}{h}\right)$$
$$=\left(1+\frac{\delta}{h}\right)y_i-\frac{\delta}{h}y_{i+1} \tag{11}$$

Substituting the equations (10) and (11) in Eq.(9), we get

$$\frac{2\varepsilon}{h}y_{i+1}-\frac{4\varepsilon}{h}y_i+\frac{2\varepsilon}{h}y_{i-1}=p_i\left[\left(1+\frac{\delta}{h}\right)y_i-\frac{\delta}{h}y_{i+1}\right]$$

$$-p_{i-1}\left[\left(1+\frac{\delta}{h}\right)y_{i-1}-\frac{\delta}{h}y_i\right]-\frac{h}{2}\left(p_i'\left(1+\frac{\delta}{h}\right)y_i-\frac{\delta}{h}y_{i+1}\right)$$

$$-\frac{h}{2}p_{i-1}'\left[\left(1+\frac{\delta}{h}\right)y_{i-1}-\frac{\delta}{h}y_i\right]+\frac{h}{2}(q_i y_i)+\frac{h}{2}(q_{i-1}y_{i-1})$$

$$+\frac{h}{2}r_i+\frac{h}{2}r_{i-1}$$

Rearrange the above equation into three recurrence relation, we get

$$E_i y_{i-1}-F_i y_i+G_i y_{i+1}=H_i\,,\ i=1,2,3......n-1 \tag{12}$$

where
$$E_i=\frac{2\varepsilon}{h}+p_{i-1}\left(1+\frac{\delta}{h}\right)+\frac{h}{2}p_{i-1}'(1+\frac{\delta}{h})-\frac{h}{2}q_{i-1}$$

$$F_i=\frac{4\varepsilon}{h}-\frac{h}{2}p_i'\left(1+\frac{\delta}{h}\right)+p_i(1+\frac{\delta}{h})+p_{i-1}\frac{\delta}{h}+\frac{\delta}{2}p_{i-1}'+\frac{h}{2}q_i$$

$$G_i=\frac{2\varepsilon}{h}+p_i\frac{\delta}{h}-\frac{\delta}{2}P_i'$$

$$H_i=\frac{h}{2}r_i+\frac{h}{2}r_{i-1}$$

3. Numerical Examples

To describe the method we consider six test examples with left and right end boundary layers.

Example 1. $\varepsilon y''(x)+y'(x-\delta)-y(x)=0$; $x\in[0,1]$ under the interval with boundary conditions y(x) = 1, $-\delta\le x\le 0$, y(1) = 1
The exact solution is given by

$$y(x)=\frac{((1\text{-}e^{m_2})e^{m_1x}+(e^{m_1}-1)e^{m_2x})}{(e^{m_1}-e^{m_2})}$$

where $m_1=\left(-1-\sqrt{1+4(\varepsilon-\delta)}\right)\big/2(\varepsilon-\delta)$ and

$m_2=\left(-1+\sqrt{1+4(\varepsilon-\delta)}\right)\big/2(\varepsilon-\delta)$.

Example 2. $\varepsilon y''(x)+0.25y'(x-\delta)-y(x)=0$ under the interval with boundary conditions *y(x)* = 1, $-\delta\le x\le 0$, y(1) = 0

Example 3. $\varepsilon y''(x)-y'(x-\delta)-y(x)=0$; under the interval with boundary conditions y(x) = 1, $-\delta\le x\le 0$, y(1) = -1
The exact solution is given by

$$y(x)=\frac{((1+e^{m_2})e^{m_1x}-(e^{m_1}+1)e^{m_2x})}{(e^{m_2}-e^{m_1})}$$

where $m_1=\left(1-\sqrt{1+4(\varepsilon+\delta)}\right)\big/2(\varepsilon+\delta)$ and

$m_2=\left(1+\sqrt{1+4(\varepsilon+\delta)}\right)\big/2(\varepsilon+\delta)$.

Example 4. $\varepsilon y''(x)-y'(x-\delta)+y(x)=0$ under the interval with boundary conditions y(x) = 1, $-\delta\le x\le 0$, y(1) = - 1

4. Discussions and Conclusions

In this paper, we have described a numerical integration technique for solving singularly perturbed delay differential equations. The second order singularly perturbed boundary value problem is transformed into an asymptotically equivalent first order neutral differential equation. Then numerical integration and linear interpolation is used to get the tri-diagonal system. Discrete invariant imbedding algorithm is used to solve this tri-diagonal system. The error analysis of the

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technique is discussed. To demonstrate the technique and affect of the delay argument in the layer, we have implemented the technique on several test examples and we have shown the layer behaviour through graphs.

From the numerical examples presented here, we observed that as δ increases, the thickness of the left end boundary layer decreases and as δ increases, the thickness of right end boundary layer increases. As the grid size h decreases, the maximum error decreases, which shows the convergence to the computed solution.

TABLE 1. THE MAXIMUM ABSOLUTE ERRORS FOR $\delta = 0.03$

ε / N	100	200	300	400	500
Example 1					
2^{-1}	1.5805e-003	7.9569e-004	5.3170e-004	3.9924e-004	3.1962e-004
2^{-2}	4.1130e-003	2.0791e-003	1.3911e-003	1.0452e-003	8.3706e-004
2^{-3}	9.1564e-003	4.6690e-003	3.1325e-003	2.3570e-003	1.8894e-003
2^{-4}	1.9899e-002	1.0335e-002	6.9813e-003	5.2710e-003	4.2338e-003
2^{-5}	4.5123e-002	2.4551e-002	1.6873e-002	1.2868e-002	1.0397e-002
Example 2					
2^{-1}	6.5308e-004	3.2800e-004	2.1900e-004	1.6437e-004	1.3156e-004
2^{-2}	1.2766e-003	6.4293e-004	4.2966e-004	3.2264e-004	2.5830e-004
2^{-3}	2.3695e-003	1.1981e-003	8.0175e-004	6.0245e-004	4.8251e-004
2^{-4}	4.2895e-003	2.1844e-003	1.4653e-003	1.1024e-003	8.8353e-004
2^{-5}	8.0121e-003	4.1358e-003	2.7884e-003	2.1028e-003	1.6880e-003
Example 3					
2^{-1}	3.7887e-003	1.9299e-003	1.2948e-003	9.7419e-004	7.8082e-004
2^{-2}	1.5347e-002	7.9592e-003	5.3719e-003	4.0537e-003	3.2550e-003
2^{-3}	3.4927e-002	1.8498e-002	1.2574e-002	9.5224e-003	7.6640e-003
2^{-4}	7.5272e-002	4.3189e-002	3.0328e-002	2.3451e-002	1.9078e-002
2^{-5}	8.0792e-002	1.2306e-001	1.4428e-001	1.5343e-001	1.5555e-001
Example 4					
2^{-1}	4.7433e-003	2.3865e-003	1.5943e-003	1.1969e-003	9.5814e-004
2^{-2}	1.0028e-002	5.0628e-003	6.4266e-003	4.8333e-003	2.0371e-003
2^{-3}	1.8863e-002	9.5867e-003	6.2857e-003	4.7533e-003	3.8732e-003
2^{-4}	3.3542e-002	1.7246e-002	1.1608e-002	8.7488e-003	7.0204e-003
2^{-5}	5.6064e-002	2.9375e-002	1.9905e-002	1.5053e-002	1.2103e-002

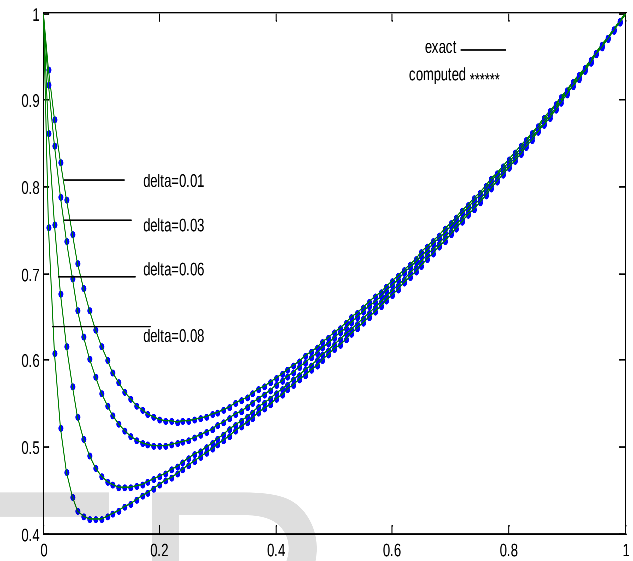

Fig.1.The numerical solution of example 1 with $\varepsilon = 0.1$

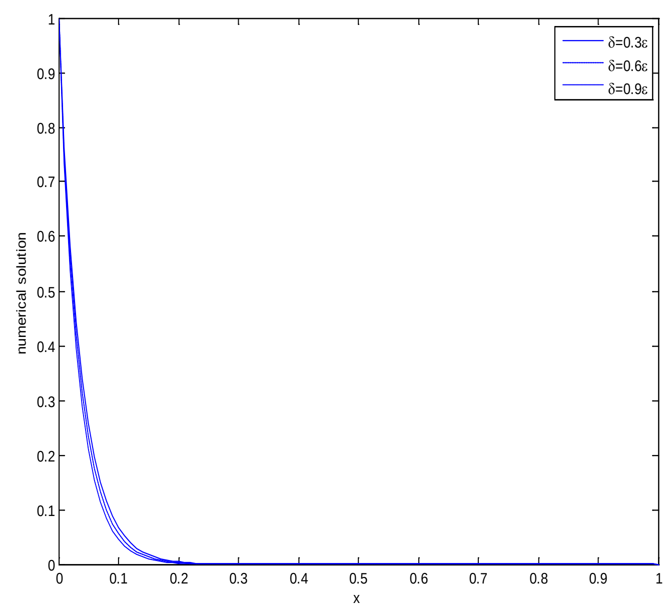

Fig.2.The numerical solution of example 2 with $\varepsilon = 0.01$

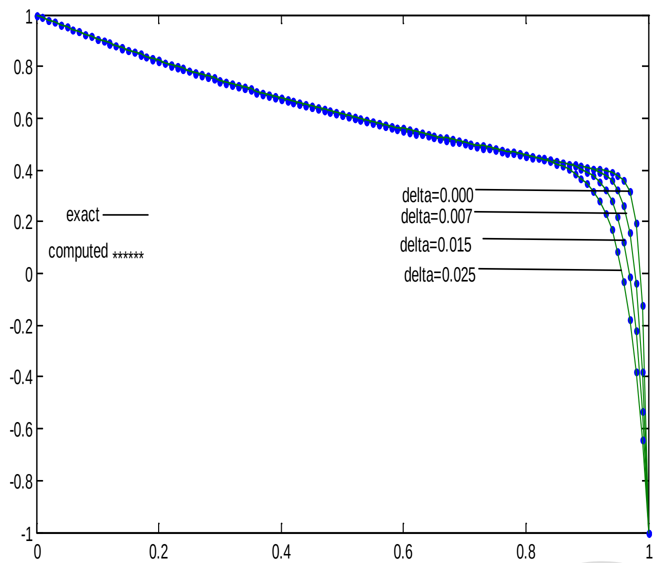

Fig 3.The numerical solution of example 3 with $\varepsilon = 0.1$

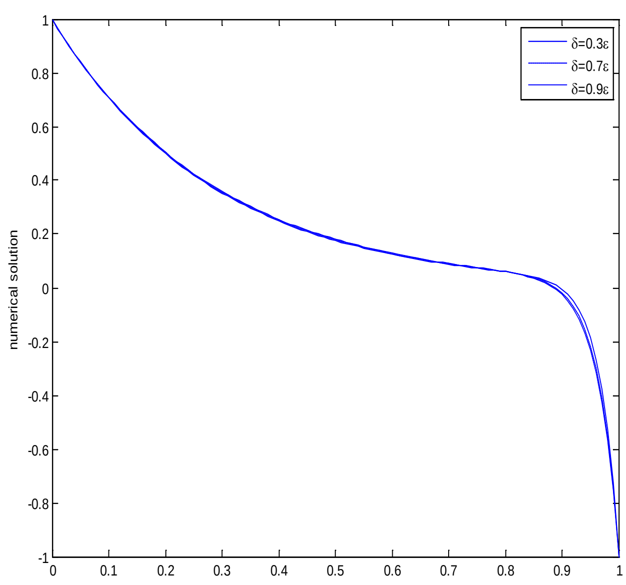

Fig 4.The numerical solution of example 4 with $\varepsilon = 0.01$

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